Linear Programming Examples

1. Given

$$\begin{aligned} \min \quad & 4x + 3y - 6z \\ \text{s.t.} \quad & y - 3z \geq 2x + 2 \\ & 3x + 2y + 5z = 10 \\ & x, z \geq 0 \end{aligned}$$

a) Convert this LP into Standard Form

Soln:

Recall that in Standard Form, we want

$$\begin{aligned} \text{Max} \quad & z = c^T x \\ \text{s.t.} \quad & a_1^T x \leq b_1 \\ & \vdots \\ & a_m^T x \leq b_m \\ & x \geq 0 \end{aligned}$$

Here are a couple of issues:

1. We're minimizing the obj func instead of maximizing it.
2. The first constraint has $\geq$ instead of $\leq$ and $2x$ is on the RHS instead of the LHS.
3. The second constraint has $=$ instead of $\leq$.
4. $y \geq 0$ is missing.

To fix #4, replace all instances of y with $y' - y''$ s.t. $y' \geq 0$, $y'' \geq 0$.

Now we have:

$$\begin{aligned} \text{Min} \quad & 4x + 3y' - 3y'' - 6z \\ \text{s.t.} \quad & y' - y'' - 3z \geq 2x + 2 \\ & 3x + 2y' - 2y'' + 5z = 10 \\ & x, y', y'', z \geq 0 \end{aligned}$$

To fix #3, change the = to LHS and RHS and then multiply the constraint that has $\geq$ by -1 to change it to $\leq$.

Now we have:

$$\begin{array}{ll} \text{Min} & 4x + 3y' - 3y'' - 6z \\ \text{s.t.} & y' - y'' - 3z \geq 2x + 2 \\ & 3x + 2y' - 2y'' + 5z \leq 10 \\ & -3x - 2y' + 2y'' - 5z \leq -10 \\ & x, y', y'', z \geq 0 \end{array}$$

To fix #2, move $2x$ to the LHS and multiply the row by -1 to change $\geq$ to $\leq$.

Now we have:

$$\begin{array}{ll} \text{Min} & 4x + 3y' - 3y'' - 6z \\ \text{s.t.} & 2x - y' + y'' + 3z \leq -2 \\ & 3x + 2y' - 2y'' + 5z \leq 10 \\ & -3x - 2y' + 2y'' - 5z \leq -10 \\ & x, y', y'', z \geq 0 \end{array}$$

Lastly, to change #1, we max $-$(obj func).

Now we have:

$$\begin{array}{ll} \text{Max} & -4x - 3y' + 3y'' + 6z \\ \text{s.t.} & 2x - y' + y'' + 3z \leq -2 \\ & 3x + 2y' - 2y'' + 5z \leq 10 \\ & -3x - 2y' + 2y'' - 5z \leq -10 \\ & x, y', y'', z \geq 0 \end{array}$$

Final Solution

b) Write the dual LP of the result from part a)

Soln:

Recall that

Primal LP		Dual LP	
max	$c^T x$	min	$b^T y$
s.t.	$Ax \le b$	s.t.	$A^T y \ge c$
	$x \ge 0$		$y \ge 0$

This was what we got in part a)

$$\begin{aligned} \text{Max} \quad & -4x - 3y' + 3y'' + 6z \\ \text{s.t.} \quad & 2x - y' + y'' + 3z \le -2 \\ & 3x + 2y' - 2y'' + 5z \le 10 \\ & -3x - 2y' + 2y'' - 5z \le -10 \\ & x, y', y'', z \ge 0 \end{aligned}$$

$$A = \begin{bmatrix} 2 & -1 & 1 & 3 \\ 3 & 2 & -2 & 5 \\ -3 & -2 & 2 & -5 \end{bmatrix}$$

$$c = \begin{bmatrix} -4 & -3 & 3 & 6 \end{bmatrix}$$

$$b = \begin{bmatrix} -2 & 10 & -10 \end{bmatrix}$$

Hence, our dual LP is:

$$\begin{aligned} \text{Min} \quad & -2y_1 + 10y_2 - 10y_3 \\ \text{s.t.} \quad & 2y_1 + 3y_2 - 3y_3 \ge -4 \\ & -y_1 + 2y_2 - 2y_3 \ge -3 \\ & y_1 - 2y_2 + 2y_3 \ge 3 \\ & 3y_1 + 5y_2 - 5y_3 \ge 6 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

Note: Since the primal LP has 3 constraints and 4 vars, the dual LP has 3 vars and 4 constraints.

2. Convert the following LP to Standard Form:

$$\begin{array}{lrl} \text{Min} & 2x_1 + 7x_2 + x_3 & \\ \text{s.t.} & x_1 - x_3 &= 7 \\ & 3x_1 + x_2 &\geq 24 \\ & x_2 &\geq 0 \\ & x_3 &\leq 0 \end{array}$$

Soln:

Let $x_1 = x_1' - x_1''$, $x_1' \geq 0$, $x_1'' \geq 0$

$$\begin{array}{lrl} \text{Max} & -2x_1' + 2x_1'' - 7x_2 - x_3 & \\ \text{s.t.} & x_1' - x_1'' - x_3 &\leq 7 \\ & -x_1' + x_1'' + x_3 &\leq -7 \\ & x_1', x_1'', x_2, -x_3 &\geq 0 \end{array}$$

3. Convert the following LP to slack form.

$$\begin{array}{lrl} \text{Max} & 2x_1 - 6x_3 & \\ \text{s.t.} & x_1 + x_2 - x_3 &\leq 7 \\ & 3x_1 - x_2 &\geq 8 \\ & -x_1 + 2x_2 + 2x_3 &\geq 0 \\ & x_1, x_2, x_3 &\geq 0 \end{array}$$

Soln:

First, convert it to standard form

$$\begin{array}{lrl} \text{Max} & 2x_1 - 6x_3 & \\ \text{s.t.} & x_1 + x_2 - x_3 &\leq 7 \\ & -3x_1 + x_2 &\leq -8 \\ & x_1 - 2x_2 - 2x_3 &\leq 0 \\ & x_1, x_2, x_3 &\geq 0 \end{array}$$

Recall:

Standard form	Slack form
Max $c^T x$	$z = c^T x$
s.t. $Ax \le b$	$s = b - Ax$
$x \ge 0$	$x, s \ge 0$

Slack form:

$$
\begin{aligned}
z &= 2x_1 - 6x_3 \\
x_4 &= 7 - x_1 - x_2 + x_3 \\
x_5 &= -8 + 3x_1 - x_2 \\
x_6 &= -x_1 + 2x_2 + 2x_3 \\
& x_1, \dots, x_6 \ge 0
\end{aligned}
$$

Basic vars: x_4, x_5, x_6

Nonbasic vars: x_1, x_2, x_3

4. Show that the following LP problem is infeasible:

$$
\begin{aligned}
\text{Max} \quad & 3x_1 - 2x_2 \\
\text{s.t.} \quad & x_1 + x_2 \le 2 \\
& -2x_1 - 2x_2 \le -10 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

Soln:

Rewriting the second constraint, we get

$x_1 + x_2 \ge 5$.

This is at odds with the first constraint.

5. Solve the following LP with the simplex algo.

$$\begin{aligned} \text{Max} \quad & 18x_1 + 12.5x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 20 \\ & x_1 \le 12 \\ & x_2 \le 16 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Soln:

First, rewrite the LP into slack form.

$$\begin{aligned} z &= 18x_1 + 12.5x_2 \\ x_3 &= 20 - x_1 - x_2 \\ x_4 &= 12 - x_1 \\ x_5 &= 16 - x_2 \\ & x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

Then, set all non-basic vars (x_1, x_2) to 0.

Next, choose a non-basic var with a positive coefficient. This will be our entering var.

I'll choose x_1. I'll find the tightest bound for x_1.

$$\begin{aligned} x_1 &= 20 - x_2 - x_3 \\ &= 20 - x_3 \\ &\le 20 \end{aligned}$$

$$\begin{aligned} x_1 &= 12 - x_4 \\ &\le 12 \leftarrow \text{Tightest bound} \end{aligned}$$

Replace all instances of x_1 with $12 - x_4$.

$$Z = 18(12 - x_4) + 12.5x_2$$
$$= 216 - 18x_4 + 12.5x_2$$

$$x_3 = 20 - (12 - x_4) - x_2$$
$$= 8 + x_4 - x_2$$

$$x_1 = 12 - x_4$$

$$x_5 = 16 - x_2$$
$$x_1, \ldots, x_5 \geq 0$$

I'll choose x_2 as the entering var and find its tightest bound.

$$x_2 = 8 + x_4 - x_3$$
$$= 8 - x_3$$

Tightest bound $\longrightarrow$ ≤ 8

Since x_4 is a non-basic var, its value is 0.

$$x_2 = 16 - x_5$$
$$\leq 16$$

Replace all instances of x_2 with $8 + x_4 - x_3$

$$Z = 216 - 18x_4 + 12.5(8 + x_4 - x_3)$$
$$= 316 - 5.5x_4 - 12.5x_3$$

$$x_1 = 12 - x_4$$
$$x_2 = 8 + x_4 - x_3$$
$$x_5 = 16 - (8 + x_4 - x_3)$$
$$= 8 - x_4 + x_3$$

Since there are no more non-basic vars, we stop.

$z = 316$
$x_1 = 12$
$x_2 = 8$
$x_3 = 0$
$x_4 = 0$
$x_5 = 8$

6. Solve the following LP using Simplex.

$$\begin{aligned} \text{Max} \quad & 5x_1 - 3x_2 \\ \text{s.t.} \quad & x_1 - x_2 \le 1 \\ & 2x_1 + x_2 \le 2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Soln:

First, write the LP in slack form.

$$\begin{aligned} z &= 5x_1 - 3x_2 \\ x_3 &= 1 - x_1 + x_2 \\ x_4 &= 2 - 2x_1 - x_2 \\ & x_1, \ldots, x_4 \ge 0 \end{aligned}$$

Then, set all non-basic vars as 0.

I'll choose x_1 as my entering var.

$$\begin{aligned} x_1 &= 1 + x_2 - x_3 \\ &= 1 - x_3 \\ &\le 1 \end{aligned}$$

$$\begin{aligned} 2x_1 &= 2 - x_2 - x_4 \\ x_1 &= 1 - \tfrac{x_4}{2} \\ &\le 1 \end{aligned}$$

We see that both eqns give us $x_1 \le 1$.
I'll choose the first eqn.

$$z = 5(1 + x_2 - x_3) - 3x_2$$
$$= 5 + 2x_2 - 5x_3$$

$$x_1 = 1 + x_2 - x_3$$
$$x_4 = 2 - 2(1 + x_2 - x_3) - x_2$$
$$= 2 - 2 - 2x_2 + 2x_3 - x_2$$
$$= 2x_3 - 3x_2$$

Now, I'll choose x_2 as my entering var.

$$x_2 = 1 - x_1 - x_3$$
$$\le 1$$

$$3x_2 = 2x_3 - x_4$$
$$= 0 - x_4$$
$$\le 0$$ ← Tightest bound

$$x_2 = \tfrac{2}{3}x_3 - \tfrac{1}{3}x_4$$

$$z = 5 + 2\left(\tfrac{2}{3}x_3 - \tfrac{1}{3}x_4\right) - 5x_3$$
$$= 5 + \tfrac{4}{3}x_3 - \tfrac{2}{3}x_4 - 5x_3$$
$$= 5 - \tfrac{11}{3}x_3 - \tfrac{2}{3}x_4$$ ← No more entering var

$$x_1 = 1 + \tfrac{1}{3}x_3 - \tfrac{1}{3}x_4$$
$$x_2 = \tfrac{2}{3}x_3 - \tfrac{1}{3}x_4$$

Final Soln:
$z = 5$
$x_1 = 1,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0$

7. You are given n jobs with a list of durations $d_1, d_2, \dots, d_n$. For every pair of jobs (i,j), you are given a boolean $P_{i,j}$.
If this is true, then job i must finish before job j can begin.
We want to find start times $S_1, S_2, \dots, S_n$ for the jobs (no job can start before time 0) s.t. the total time to complete all jobs is minimized while ensuring that the pre-requisite constraints are met. Write a LP to solve this problem.

Soln:
I'll let S_i denote the start time of job i.
I'll let T be an upper bound on the time completion.

Our LP:

$$
\begin{aligned}
\text{Min} \quad & T \\
\text{s.t.} \quad & \sum_i S_i + d_i \leq T \\
& d_i + S_i \leq S_j, \ i \neq j \text{ and } P_{ij} = 1 \text{ for } i,j \in \{1, \dots, n\} \\
& S_i \geq 0, \text{ for } i \in \{1, \dots, n\}
\end{aligned}
$$

8. Suppose you're writing down a binary integer linear program. You have 3 vars x, y, and z and the constraint $x, y, z \in \{0,1\}$. For each relationship below, write a LP to solve it.

a) $z = x \wedge y$. We want $z=1$ when x and $y=1$ and 0 otherwise

Soln:

$z \leq x$

$z \leq y$

$z \geq x+y-1$

b) $z = x \vee y$

Soln:

$z \geq x$

$z \geq y$

$z \leq x+y$

c) $z = \neg x$ (Not x)

Soln:

$z = 1-x$

9. A farmer has 110 hectares piece of land. He plans to grow wheat and barley. He has data regarding the cost, profit and man-days for growing each variety (shown below)

Variety	Cost (Price / Hec)	Net Profit (Price/Hec)	Man-days/Hec
Wheat	100	50	10
Barley	200	120	30

The farmer has a budget of \$10,000 and 1200 man-days. Find the opt value and opt soln.

Soln:

Let x be the total amount of hecs used to grow wheat.

Let y be the total amount of hecs used to grow barley.

LP:

$$\begin{aligned} \text{Max} \quad & 50x + 120y \\ \text{s.t.} \quad & 100x + 200y \le 10{,}000 \rightarrow x + 2y \le 100 \\ & 10x + 30y \le 1{,}200 \rightarrow x + 3y \le 120 \\ & x + y \le 110 \\ & x, y \ge 0 \end{aligned}$$

I'll use the simplex algo to solve it.

First, I'll change it to slack form.

$$\begin{aligned} z &= 50x + 120y \\ s_1 &= 100 - x - 2y \\ s_2 &= 120 - x - 3y \\ s_3 &= 110 - x - y \\ & x, y, s_1, s_2, s_3 \ge 0 \end{aligned}$$

I'll pick x as my entering var.

$x = 100 - 2y - s_1$	$x = 120 - 3y - s_2$	$x = 110 - y - s_3$
Tightest Bound → ≤ 100	≤ 120	≤ 110

$$x = 100 - 2y - s_1$$

Now, we have

$$\begin{aligned} z &= 5000 + 20y - 50s_1 \\ x &= 100 - 2y - s_1 \\ s_2 &= 20 - y + s_1 \\ s_3 &= 10 + y + s_1 \\ & x, y, s_1, s_2, s_3 \ge 0 \end{aligned}$$

Now, I'll pick y as my entering var.

$y = 50 - \frac{1}{2}x$ | $y = 20 + s_1 - s_2$ | $y = s_3 - 10 - s_1$

≤ 50 | ≤ 20 | Unusable

Tightest Bound

$y = 20 + s_1 - s_2$

Now we have

$Z = 5400 - 30s_1 - 20s_2$

$x = 100 - 2(20 + s_1 - s_2) - s_1$

$\quad = 60 - 3s_1 + 2s_2$

$y = 20 + s_1 - s_2$

$s_3 = 10 - (20 + s_1 - s_2) + s_1$

$\quad = -10 + s_2$

$x, y, s_1, s_2, s_3 \geq 0$

Max profit: \$5400

Should produce wheat in 60 hecs.

Should produce barley in 20 hecs.